647510 **MSC4MatC402x** **Seat No :________**

M.Sc.(Maths.) Semester - 4 (*CBCS*) Examination
April/May-2023 (NEW COURSE)
Integration Theory (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(A) Answer the following questions. (Any Two) **(10)**

1. Show that every $\sigma -$ finite measure space is saturated.
2. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on a measure space $(X, \mathcal{A}, \mu)$, then show that $\int_X \left(\lim_n \inf f_n\right) d\mu \leq \lim_n \inf \int_X f_n \, d\mu$.
3. Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathcal{A}, \mu)$ converging to a function f(pointwise) on X. Let $E \in \mathcal{A}$. If g is integrable over E and $|f_n| \leq g$ on E for all n, then show that $\int_E f \, d\mu = \lim_n \int_E f_n \, d\mu$.

Q.1(B) Answer the following question. (Any One) **(04)**

1. Let μ be a measure on a measurable space $(X, \mathcal{A})$. If E and F are measurable subsets of X, $E \subset F$ and if $\mu(F) < \infty$, then show that $\mu(F - E) = \mu(F) - \mu(E)$.
2. Give an example of measure space $(X, \mathcal{A}, \mu)$ which is not complete and give appropriate reason.

Q.2(A) Answer the following questions. (Any Two) **(10)**

1. Let v be a signed measure on a measurable space $(X, \mathcal{A})$, and let E be a measurable subset of X such that $0 < v(E) < \infty$. Show that E contains a positive set F such that $v(F) > 0$.
2. Let v be a signed measure on a measurable space $(X, \mathcal{A})$, then show that there exist a positive set A and negative set B such that $X = A \cup B$ and $A \cap B = \emptyset$.
3. Let v be a signed measure on a measurable space $(X, \mathcal{A})$, then show that the Jordan decomposition of v is unique.

Q.2(B) Answer the following question. (Any One) **(04)**

1. Let v be a signed measure on a measurable space $(X, \mathcal{A})$, and let A and B be positive sets. Then show that $A \cup B$ is a positive set.
2. Let v_1 and v_2 be finite signed measures on a measurable space $(X, \mathcal{A})$, then show that $|v_1 + v_2| \leq |v_1| + |v_2|$.

Q.3(A) Answer the following questions. (Any Two) **(10)**

1. Let v_1 and v_2 be measures and μ be a $\sigma-$ finite measure on a measurable space $(X, \mathcal{A})$, and let $v_1 \ll \mu$ and $v_2 \ll \mu$. Then show that $\left[\frac{d(v_1+v_2)}{d\mu}\right] = \left[\frac{dv_1}{d\mu}\right] + \left[\frac{dv_2}{d\mu}\right]$.
2. Let v and μ be a $\sigma-$ finite measure on a measurable space $(X, \mathcal{A})$, then show that there exist a pair of measures v_0 and v_1 such that $v_0 \perp \mu$, $v_1 \ll \mu$ and $v = v_0 + v_1$.
3. Let v be a signed measure and μ be a $\sigma-$ finite measure on a measurable space $(X, \mathcal{A})$, and let $v \ll \mu$. If f is nonnegative measurable function on X, then show that $\int_E f\, dv = \int_E f \left[\frac{dv}{d\mu}\right] d\mu$ for every $E \in \mathcal{A}$.

Q.3(B) Answer the following question. (Any One) **(04)**

1. Let v_1 and v_2 be finite signed measure on a measurable space $(X, \mathcal{A})$, and μ be a measure on $(X, \mathcal{A})$. If $v_1 \ll \mu$ and $v_2 \ll \mu$, then show that $v_1 + v_2 \ll \mu$.
2. Let v be a signed measure and μ be a measure on a measurable space $(X, \mathcal{A})$. If $v \perp \mu$ and $v \ll \mu$, then show that $v = 0$.

Q.4(A) Answer the following questions. (Any Two) **(10)**

1. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \le p < \infty$. If $f, g \in L^p(\mu)$, then show that $\|f + g\|_p \le \|f\|_p + \|g\|_p$.
2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $1 \le p < r < \infty$. Then show that $L^r(\mu) \subset L^p(\mu)$.
3. Let $1 \le p < \infty$. Let $f \in L^p(\mu)$, and let $\epsilon > 0$. Then show that there is a measurable simple function s vanishing outside a set of finite measure such that $\|f - s\|_p < \epsilon$.

Q.4(B) Answer the following question. (Any One) **(04)**

1. Let $(X, \mathcal{A}, \mu)$ be a measure space, and a measurable function f is bounded a.e. $[\mu]$ on X. Then show that f is finite a.e. $[\mu]$ on X.
2. Define Baire measure on a real line.

Q.5(A) Answer the following questions. (Any Two) **(10)**

1. Let μ be a finite measure defined on a $\sigma-$ algebra M which contains all the Baire sets of a locally compact space X. If μ is inner regular, then show that it is regular.
2. Let μ^* be a topologically regular outer measure on a locally compact Hausdorff space X. Show that each Borel sets is μ^*- measurable.
3. Let μ be a Baire measure on a locally compact space X, and let E be a $\sigma-$ bounded Baire set in X. If $\epsilon > 0$, then show that there is a $\sigma-$ compact open set O containing E such that $\mu(O - E) < \epsilon$.

Q.5(B) Answer the following question. (Any One) **(04)**

1. State Riesz-Markov theorem.
2. Let X be a locally compact Hausdorff space, and let M be a $\sigma-$ algebra on X containing all Baire sets. When is $E \in M$ called outer regular for μ ?

****